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BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC EQUATIONS WITH A CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this paper we study local and nonlocal boundary value problems for the hyperbolic equations of the general form with variable coefficients and with a Caputo fractional derivative. For the investigation the posed problem, one functional space of fractional order is introduced. The posed problem is reduced to the integral equation and the existence of its solution is proved by the help of a priori estimate.

Keywords: hyperbolic differential equation, fractional derivative, Riemann -Liouville integral, Caputo derivative, nonlocal problem.

AMS Subject Classification: 26A33, 35R11, 34K37.

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1 Introduction

The fractional calculus deals with extensions of derivatives and integrals to noninteger orders. The field of fractional differential equations has been subjected to an intensive development of the theory and applications in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering (Baleanu et al., 2012; Hilfer, 2000; Mainardi, 2010; Miller & Ross, 1993; Nakhushev, 2000; Oldham & Spanier, 1974; Samko et al., 1993; Tarasov, 2011; Vázquez et al., 2011). In recent years, several qualitative results for ordinary and partial fractional differential equations have been obtained (see (Agarwal et al., 2017; Dhiman et al., 2017; Diethelm, 2010; Gorenflo et al., 2015; Kilbas & Marzan, 2004; Kilbas et al., 2006; Kochubei, 2013; Kubica& Ryszewska, 2018; Liang et al., 2018; Podlubny, 1999; Zhou, 2014) and references therein).

In the simplest physical processes, there arise fractional derivatives with respect to two independent variables– the coordinate and time. Therefore, the study of the boundary value problems for the fractional partial differential equations is a topical field of fractional calculus. The Darboux problem for fractional partial hyperbolic differential equations was studied in (Abbas et al., 2012; Vityuk & Mykhailenko, 2008, 2011).

Nonlocal boundary value problems are usually called problems with given conditions that connect the values of the desired solution and/or its derivatives either at different points of the boundary or at boundary points and some interior points. Note that nonlocal problems for hyperbolic differential equations and the corresponding optimal control problems are being actively studied at present time (Byszewski, 1991; Yusubov, 2014, 2017). But, nonlocal problems for the hyperbolic equations of fractional order are less investigated (Abbas & Benchohra, 2009).

In this paper we investigate the existence and uniqueness of the solutions of the following fractional hyperbolic differential equations with boundary value conditions:

$$\begin{aligned}&(l_{11}u)(x) \equiv D_1^{\alpha_1} D_2^{\alpha_2} u + a_1(x) D_1^{\alpha_1} u + a_2(x) D_2^{\alpha_2} u + a_3(x) u = \varphi_{11}(x),\\ &x \in D = [0, X_1] \times [0, X_2]\,,\end{aligned} \tag{1}$$

$$\begin{cases} (l_1 u)(x_1) \equiv \alpha_1 D_1^{\alpha_1} u(x_1, 0) + \beta_1 D_1^{\alpha_1} u(x_1, X_2) = \varphi_1(x_1), \\ \qquad\qquad x_1 \in J_{x_1} = [0,\ X_1]\,, \\ (l_2 u)(x_2) \equiv \alpha_2 D_2^{\alpha_2} u(0, x_2) + \beta_2 D_2^{\alpha_2} u(X_1, x_2) = \varphi_2(x_2), \\ \qquad\qquad x_2 \in J_{x_2} = [0,\ X_2]\,, \\ l_0 u \equiv u(0,0) = \varphi_0, \end{cases} \tag{2}$$

where $X_1, X_2 > 0$, $D_1^{\alpha_1} D_2^{\alpha_2}$ is the mixed Caputo fractional derivative of order $\alpha = (\alpha_1, \alpha_2) \in (0,1] \times (0,1]$, $D_i^{\alpha_i}$, $i = 1,\ 2$ are the partial α_i -order Caputo derivatives.

It is further assumed that the functions $a_1(x)$, $a_2(x)$, $a_3(x)$ and $\varphi_{11}(x)$ are continuous in D, the functions $\varphi_1(x_1)$ and $\varphi_2(x_2)$ continuous in J_{x_1} and J_{x_2}, respectively, α_i, β_i, $i = 1, 2$ are real numbers.

The posed problem is reduced to the integral equation and the existence of its solution is proved by the help of a priori estimates.

This paper is organized as follows. In Section 2, we recall briefly some basic definitions and preliminary facts which will be used throughout the following section. The posed problem is reduced to the integral equation in Section 3. The existence and uniqueness results for problem (1), (2) are obtained in Section 4.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(D)$ we denote the Banach space of all continuous functions from D into R with the norm

$$\|u\|_{C(D)} = \max_{x \in D} |u(x)|\,.$$

$L^1(D)$ is the space of Lebesgue –integrable functions $u : D \to R$ with the norm

$$\|u\| = \int_0^{X_1} \int_0^{X_2} |u(x_1, x_2)|\, dx_1 dx_2.$$

Definition 1. *(Kilbas et al., 2006; Samko et al., 1993). Let $\alpha_1 \in (0, \infty)$, and $u \in L^1(D)$. The partial Riemann-Liouville integral of order α_1 of $u(x)$ with respect to x_1 is defined by the expression*

$$\left(I_{0,x_1}^{\alpha_1} u\right)(x_1, x_2) = \frac{1}{\Gamma(\alpha_1)} \int_0^{x_1} (x_1 - s_1)^{\alpha_1 - 1} u(s_1, x_2) ds_1,$$

for almost all $x_1 \in [0, X_1]$ and almost all $x_2 \in [0, X_2]$, where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined as

$$\Gamma(\xi) = \int_0^{\infty} t^{\xi - 1} e^{-t} dt, \xi > 0.$$

Analogously, we define the integral

$$\left(I_{0,x_2}^{\alpha_2}u\right)(x_1,x_2)=\frac{1}{\Gamma(\alpha_2)}\int\limits_0^{x_2}(x_2-s_2)^{\alpha_2-1}u(x_1,s_2)ds_2,$$

for almost all $x_1\in[0,X_1]$ and almost all $x_2\in[0,X_2]$.

Definition 2. *(Kilbas et al., 2006; Samko et al., 1993). Let $\alpha_1\in(0,1]$ and $u\in L^1(D)$. The Riemann-Liouville fractional derivative of order α_1 of $u(x_1,x_2)$ with respect to x_1 is defined by the expression*

$$\left(D_{0,x_1}^{\alpha_1}u\right)(x_1,x_2)=\frac{\partial}{\partial x_1}\left(I_{0,x_1}^{1-\alpha_1}u\right)(x_1,x_2),$$

for almost all $x_1\in[0,X_1]$ and almost all $x_2\in[0,X_2]$.

Analogously, we define the derivative

$$\left(D_{0,x_2}^{\alpha_2}u\right)(x_1,x_2)=\frac{\partial}{\partial x_2}\left(I_{0,x_2}^{1-\alpha_2}u\right)(x_1,x_2),$$

for almost all $x_1\in[0,X_1]$ and almost all $x_2\in[0,X_2]$.

Definition 3. *(Vityuk & Mykhailenko, 2011). Let $\alpha_1\in(0,1]$ and $u\in C(D)$. The Caputo fractional derivative (regularized derivative) of order α_1 of $u(x_1,x_2)$ with respect to x_1 is defined by the expression*

$$\left(D_1^{\alpha_1}u\right)(x_1,x_2)\equiv\left({}^C D_{0,x_1}^{\alpha_1}u\right)(x_1,x_2)=\frac{\partial}{\partial x_1}\left(I_{0,x_1}^{1-\alpha_1}(u(s_1,x_2)-u(0,x_2))\right)(x_1,x_2)$$

$$=\frac{1}{\Gamma(1-\alpha_1)}\frac{\partial}{\partial x_1}\int\limits_0^{x_1}(x_1-s_1)^{-\alpha_1}(u(s_1,x_2)-u(0,x_2))ds_1.$$

Analogously, we define the derivative

$$\left(D_2^{\alpha_2}u\right)(x_1,x_2)\equiv\left({}^C D_{0,x_2}^{\alpha_2}u\right)(x_1,x_2)=\frac{\partial}{\partial x_2}\left(I_{0,x_2}^{1-\alpha_2}(u(x_1,s_2)-u(x_1,0))\right)(x_1,x_2)$$

$$=\frac{1}{\Gamma(1-\alpha_2)}\frac{\partial}{\partial x_2}\int\limits_0^{x_2}(x_2-s_2)^{-\alpha_2}(u(x_1,s_2)-u(x_1,0))ds_2.$$

Definition 4. *(Kilbas et al., 2006; Samko et al., 1993). Let $\alpha=(\alpha_1,\alpha_2)$, $\alpha_i>0$, $i=1,\ 2$, $\theta=(0,\ 0)$ and $u\in L^1(D)$. The left-sided mixed Riemann-Liouville integral of order α of $u(x_1,x_2)$ is defined by the expression*

$$\left(I_{\theta,x}^{\alpha}u\right)(x)=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int\limits_0^{x_1}\int\limits_0^{x_2}(x_1-s_1)^{\alpha_1-1}(x_2-s_2)^{\alpha_2-1}u(s_1,s_2)ds_1ds_2.$$

In particular,

$$\left(I_{\theta,x}^{\theta}u\right)(x)=u(x),\quad \left(I_{\theta,x}^{\sigma}u\right)(x)=\int\limits_0^{x_1}\int\limits_0^{x_2}u(s_1,s_2)ds_1ds_2,$$

for almost all $x\in D$, where $\sigma=(1,1)$.

For instance, $I^{\alpha}_{\theta,x}u$ *exists for all* $\alpha_i > 0$, $i = 1, 2$, *when* $u \in L^1(D)$. *Note also that when* $u \in C(D)$, *then* $\left(I^{\alpha}_{\theta,x}u\right)(x) \in C(D)$, *moreover*

$$\left(I^{\alpha}_{\theta,x}u\right)(x_1,0) = \left(I^{\alpha}_{\theta,x}u\right)(0,x_2) = 0, x_i \in [0, X_i], i = 1, 2.$$

By $1-\alpha$ *we mean* $(1-\alpha_1, 1-\alpha_2) \in (0,1] \times (0,1]$. *We denote by* $D_1D_2 := \frac{\partial^2}{\partial x_1 \partial x_2}$, *the mixed second order partial derivative.*

Definition 5. *(Vityuk & Mykhailenko, 2011). Let* $\alpha \in (0,1] \times (0,1]$ *and* $u \in C(D)$. *The mixed fractional Riemann-Liouville derivative of order* α *of* $u(x)$ *is defined by the expression* $\left(D^{\alpha}_{\theta,x}u\right)(x) = \left(D_1D_2I^{1-\alpha}_{\theta,x}u\right)(x)$ *and the mixed Caputo fractional derivative (mixed regularized derivative) of order* α *of a function* $u(x)$ *is defined by the expression*

$$(D^{\alpha}u)(x) = (D_1^{\alpha_1}D_2^{\alpha_2}u)(x_1,x_2) = \left({}^{C}D^{\alpha}_{\theta,x}u\right)(x_1,x_2) = D_1D_2\times$$

$$\times\left(I^{1-\alpha}_{\theta,x}\left[u(s_1,s_2) - u(s_1,0) - u(0,s_2) + u(0,0)\right]\right)(x_1,x_2) = \frac{1}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}\times$$

$$\times\frac{\partial^2}{\partial x_1 \partial x_2}\int_0^{x_1}\int_0^{x_2}(x_1-s_1)^{-\alpha_1}(x_2-s_2)^{-\alpha_2}(u(s_1,s_2) - u(s_1,0) - u(0,s_2) + u(0,0))ds_1ds_2.$$

3 Reduction of Problem (1), (2) to the Equivalent Integral Equation

We define the functional spaces as follows

$$C^{\alpha}(D) = \left\{u(x) \in C(D) | D_1^{\alpha_1}u(x) \in C(D),\ \ D_2^{\alpha_2}u(x) \in C(D),\ \ D^{\alpha}u(x) \in C(D)\right\}.$$

Lemma 1. *The space* $C^{\alpha}(D)$ *endowed with the norm*

$$\|u\|_{C^{\alpha}(D)} = \|u\|_{C(D)} + \|D_1^{\alpha_1}u\|_{C(D)} + \|D_2^{\alpha_2}u\|_{C(D)} + \|D^{\alpha}u\|_{C(D)} \tag{3}$$

is a Banach space.

We consider also the space

$$H^{\alpha} = C(D) \times C(J_{x_1}) \times C(J_{x_2}) \times R$$

of elements $b = (b_{11}, b_1, b_2, b_0)$ *with the norm*

$$\|b\|_{H^{\alpha}} = \|b_{11}\|_{C(D)} + \|b_1\|_{C(J_{x_1})} + \|b_2\|_{C(J_{x_2})} + \|b_0\|_R,$$

where R *is a space of real numbers.*

The structural characteristic of the space $C^{\alpha}(D)$ is given bellow.

Theorem 1. *The space $C^{\alpha}(D)$ consists of those and only those functions $u(x) \in C(D)$ which can be represented in the form*

$$u(x) = (Q\,b)\,(x) \equiv b_0 + \left(I_{0,x_1}^{\alpha_1} b_1\right)\,(x_1) + \left(I_{0,x_2}^{\alpha_2} b_2\right)\,(x_2) + \left(I_{\theta,x}^{\alpha} b_{11}\right)\,(x) \qquad (4)$$

where $b_{11}(x) \in C(D)$, $b_1(x_1) \in C(J_{x_1})$, $b_2 \in C(J_{x_2})$, $b_0 \in R$,

$$\left(I_{0,x_1}^{\alpha_1} b_1\right)\,(x_1) = \frac{1}{\Gamma(\alpha_1)} \int\limits_0^{x_1} (x_1 - s_1)^{\alpha_1 - 1} b_1(s_1) ds_1,$$

$$\left(I_{0,x_2}^{\alpha_2} b_2\right)\,(x_2) = \frac{1}{\Gamma(\alpha_2)} \int\limits_0^{x_2} (x_2 - s_2)^{\alpha_2 - 1} b_2(s_2) ds_2,$$

$$\left(I_{\theta,x}^{\alpha} b_{11}\right)\,(x) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int\limits_0^{x_1} \int\limits_0^{x_2} (x_1 - s_1)^{\alpha_1 - 1} (x_2 - s_2)^{\alpha_2 - 1} b_{11}(s_1, s_2) ds_1 ds_2.$$

It follows from (4) that

$$b_{11}(x) = (D^{\alpha} u)\,(x), b_1(x_1) = (D_1^{\alpha_1} u)\,(x_1, 0), b_2(x_2) = (D_2^{\alpha_2} u)\,(0, x_2), b_0 = u(0,0).$$

Lemma 2. *The operator Q defined by (4) satisfies*

$$C_1 \left\| b \right\|_{H^{\alpha}} \leq \left\| Q\,b \right\|_{C^{\alpha}(D)} \leq C_2 \left\| b \right\|_{H^{\alpha}}, \forall b \in H^{\alpha},$$

where $C_i = const > 0$, $i = 1,\ 2$.

The number b_0 and the functions $b_1(x_1)$, $b_2(x_2)$, $b_{11}(x)$ on the right-hand side of (4) are independent elements of the function $u(x) \in C^{\alpha}(D)$. These assertions show that a linear homeomorphism between $C^{\alpha}(D)$ and H^{α} exists. That is, the space $C^{\alpha}(D)$ has the isomorphic decomposition $C^{\alpha}(D) = C(D) \times C(J_{x_1}) \times C(J_{x_2}) \times R = H^{\alpha}$. This important property of (4) can be used for investigation of the initial boundary-value problems in $C^{\alpha}(D)$.

We seek a solution of problem (1), (2) in the space $C^{\alpha}(D)$ with dominating mixed Caputo derivative $D_1^{\alpha_1} D_2^{\alpha_2} u$ and with the norm (3). Problem (1), (2) is a linear nonhomogeneous problem. We consider the problem as an operator equation

$$l\,u = \varphi, \qquad (5)$$

with linear operator $l = (l_{11},\, l_1,\, l_2,\, l_0)$ and $\varphi = (\varphi_{11}(x),\, \varphi_1(x_1),\, \varphi_2(x_2),\, \varphi_0)$.

The existing conditions guarantee that operator l is bounded from $C^{\alpha}(D)$ into the Banach space H^{α}. If, for the given $\varphi \in H^{\alpha}$, problem (1), (2) has a unique solution $u \in C^{\alpha}(D)$ with $\left\| u \right\|_{C^{\alpha}(D)} \leq k \left\| \varphi \right\|_{H^{\alpha}}$, then (1), (2) is a well-posed problem, where k is an independent of φ constant. We observe that the considered problem is a well-posed one if and only if the operator l is a homeomorphism (linear) between $C^{\alpha}(D)$ and H^{α}.

Equation (5) can be reduced to the equivalent equation

$$l\,Q\,b = \varphi \qquad (6)$$

with the unknown $b = (b_{11}(x),\, b_1(x_1),\, b_2(x_2),\, b_0) \in H^{\alpha}$ by the transformation $u = Q\,b$.

We choose the element

$$b = (b_{11}(x),\, b_1(x_1),\, b_2(x_2),\, b_0) \in H^{\alpha}$$

such that the corresponding function $u(x)$, defined by the representation (4) would satisfy the conditions (2).

For this purpose substituting (4) into (2), we obtain

$$(l_1Qb)(x_1) \equiv \alpha_1 b_1(x_1) + \beta_1 \left[b_1(x_1) + \tfrac{1}{\Gamma(\alpha_2)} \int_0^{X_2} (X_2 - s_2)^{\alpha_2 - 1} b_{11}(x_1, s_2) ds_2 \right] = \varphi_1(x_1),$$
$$(l_2Qb)(x_2) \equiv \alpha_2 b_2(x_2) + \beta_2 \left[b_2(x_2) + \tfrac{1}{\Gamma(\alpha_1)} \int_0^{X_1} (X_1 - s_1)^{\alpha_1 - 1} b_{11}(s_1, x_2) ds_1 \right] = \varphi_2(x_2),$$

$$l_0 Qb \equiv b_0 = \varphi_0. \tag{7}$$

Let $\alpha_i + \beta_i \neq 0,\ i = 1,\ 2$, then from the first and second formulas of (7), correspondingly, we have

$$\begin{aligned} b_1(x_1) &= \tfrac{1}{\alpha_1+\beta_1} \left[\varphi_1(x_1) - \tfrac{\beta_1}{\Gamma(\alpha_2)} \int_0^{X_2} (X_2 - s_2)^{\alpha_2-1} b_{11}(x_1,\ s_2) ds_2 \right], \\ b_2(x_2) &= \tfrac{1}{\alpha_2+\beta_2} \left[\varphi_2(x_2) - \tfrac{\beta_2}{\Gamma(\alpha_1)} \int_0^{X_1} (X_1 - s_1)^{\alpha_1-1} b_{11}(s_1, x_2) ds_1 \right]. \end{aligned} \tag{8}$$

Substituting (8) and the third formula of (7) into (4), we obtain that the arbitrary function $u \in C^{\alpha}(D)$ can be represented as

$$u(x) = \varphi_0 + \frac{1}{(\alpha_1+\beta_1)\Gamma(\alpha_1)} \int_0^{x_1} (x_1 - s_1)^{\alpha_1-1} \varphi_1(s_1) ds_1 + \frac{1}{(\alpha_2+\beta_2)\Gamma(\alpha_2)} \int_0^{x_2} (x_2 - s_2)^{\alpha_2-1} \varphi_2(s_2) ds_2$$
$$- \frac{\beta_1}{(\alpha_1+\beta_1)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1}\int_0^{X_2} (x_1 - s_1)^{\alpha_1-1} (X_2 - s_2)^{\alpha_2-1} b_{11}(s_1, s_2) ds_1 ds_2$$
$$- \frac{\beta_2}{(\alpha_2+\beta_2)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_2}\int_0^{X_1} (X_1 - s_1)^{\alpha_1-1} (x_2 - s_2)^{\alpha_2-1} b_{11}(s_1, s_2) ds_1 ds_2$$
$$+ \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1}\int_0^{x_2} (x_1 - s_1)^{\alpha_1-1} (x_2 - s_2)^{\alpha_2-1} b_{11}(s_1, s_2) ds_1 ds_2. \tag{9}$$

Immediaty calculations give

$$D_1^{\alpha_1} u(x) = \tfrac{\varphi_1(x_1)}{\alpha_1+\beta_1} - \tfrac{\beta_1}{(\alpha_1+\beta_1)\Gamma(\alpha_2)} \int_0^{X_2} (X_2 - s_2)^{\alpha_2-1} b_{11}(x_1, s_2) ds_2$$
$$+ \tfrac{1}{\Gamma(\alpha_2)} \int_0^{x_2} (x_2 - s_2)^{\alpha_2-1} b_{11}(x_1, s_2) ds_2,$$

$$D_2^{\alpha_2} u(x) = \frac{\varphi_2(x_2)}{\alpha_2+\beta_2} - \frac{1}{(\alpha_2+\beta_2)\Gamma(\alpha_1)} \int_0^{X_1} (X_1 - s_1)^{\alpha_1-1} b_{11}(s_1, x_2) ds_1 \tag{10}$$

$$+ \tfrac{1}{\Gamma(\alpha_1)} \int_0^{x_1} (x_1 - s_1)^{\alpha_1-1} b_{11}(s_1, x_2) ds_1,$$
$$D^{\alpha_1\alpha_2} u(x) = b_{11}(x).$$

Substituting (9) and (10) into (1), we obtain that equation (1) is equivalent to the integral equation

$$(l_{11}Qb)(x) \equiv b_{11}(x) + (Ab_{11})(x) + (Bb_{11})(x) = \Phi(x), \tag{11}$$

where

$$\begin{aligned}
&(Ab_{11})(x) \equiv \tfrac{a_1(x)}{\Gamma(\alpha_2)} \int\limits_0^{x_2} (x_2 - s_2)^{\alpha_2 - 1} b_{11}(x_1, s_2) ds_2 \\
&+ \tfrac{a_2(x)}{\Gamma(\alpha_1)} \int\limits_0^{x_1} (x_1 - s_1)^{\alpha_1 - 1} b_{11}(s_1, x_2) ds_1 \\
&+ \tfrac{a_3(x)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int\limits_0^{x_1} \int\limits_0^{x_2} (x_1 - s_1)^{\alpha_1 - 1} (x_2 - s_2)^{\alpha_2 - 1} b_{11}(s_1, s_2) ds_1 ds_2,
\end{aligned} \tag{12}$$

$$\begin{aligned}
&(Bb_{11})(x) \equiv -\tfrac{\beta_1 a_1(x)}{(\alpha_1+\beta_1)\Gamma(\alpha_2)} \int\limits_0^{X_2} (X_2 - s_2)^{\alpha_2 - 1} b_{11}(x_1, s_2) ds_2 \\
&- \tfrac{\beta_2 a_2(x)}{(\alpha_2+\beta_2)\Gamma(\alpha_1)} \int\limits_0^{X_1} (X_1 - s_1)^{\alpha_1 - 1} b_{11}(s_1, x_2) ds_1 \\
&- \tfrac{\beta_1 a_3(x)}{(\alpha_1+\beta_1)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int\limits_0^{x_1} \int\limits_0^{X_2} (x_1 - s_1)^{\alpha_1 - 1} (X_2 - s_2)^{\alpha_2 - 1} b_{11}(s_1, s_2) ds_1 ds_2 \\
&- \tfrac{\beta_2 a_3(x)}{(\alpha_2+\beta_2)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int\limits_0^{x_2} \int\limits_0^{X_1} (X_1 - s_1)^{\alpha_1 - 1} (x_2 - s_2)^{\alpha_2 - 1} b_{11}(s_1, s_2) ds_1 ds_2,
\end{aligned} \tag{13}$$

$$\begin{aligned}
\Phi(x) = \varphi_{11}(x) - a_1(x)\frac{\varphi_1(x_1)}{\alpha_1 + \beta_1} - a_2(x)\frac{\varphi_2(x_2)}{\alpha_2 + \beta_2} - a_3(x)\Bigg[\varphi_0 + \frac{1}{(\alpha_1 + \beta_1)\Gamma(\alpha_1)} \\
\times \int\limits_0^{x_1} (x_1 - s_1)^{\alpha_1 - 1} \varphi_1(s_1) ds_1 + \frac{1}{(\alpha_2 + \beta_2)\Gamma(\alpha_2)} \int\limits_0^{x_2} (x_2 - s_2)^{\alpha_2 - 1} \varphi_2(s_2) ds_2\Bigg].
\end{aligned} \tag{14}$$

Thus, the following theorem is proved.

Theorem 2. *Assume that $\alpha_i + \beta_i \neq 0$, $i = 1, 2$. Then in order that the operator $l = (l_{11}, l_1, l_2, l_0)$ of problem (1), (2) would be a homeomorphism between the spaces $C^\alpha(D)$ and H^α,it is necessary and sufficient that integral equation (11) has unique solution $b_{11}(x) \in C(D)$ for any $\Phi(x) \in C(D)$.*

4 Existence and Uniqueness the Solutions of the Problem (1), (2)

We see that, the problem of finding the solution $u(x) \in C^\alpha(D)$ of (1), (2) is equivalent to the problem of finding the solution $b_{11}(x) \in C(D)$ of integral equation (11). The conditions imposed on the coefficients indicate that the operator $l_{11}Q: \ C(D) \to C(D)$ is bounded.

We rewrite integral equations (11) in the form

$$(I + A)b_{11} + Bb_{11} = \Phi. \tag{15}$$

Now we introduce the following operators

$$(A_1 b_{11})(x) = \frac{a_1(x)}{\Gamma(\alpha_2)} \int\limits_0^{x_2} (x_2 - s_2)^{\alpha_2 - 1} b_{11}(x_1, s_2) ds_2,$$

$$(A_2b_{11})(x) = \frac{a_2(x)}{\Gamma(\alpha_1)} \int_0^{x_1} (x_1 - s_1)^{\alpha_1-1} b_{11}(s_1, x_2) ds_1,$$

$$(A_3b_{11})(x) = \frac{a_3(x)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1}\int_0^{x_2} (x_1 - s_1)^{\alpha_1-1}(x_2 - s_2)^{\alpha_2-1} b_{11}(s_1, s_2) ds_1 ds_2,$$

$$(\tilde{A}_1b_{11})(x) = \frac{\|a_1\|_{C(D)}}{\Gamma(\alpha_2)} \int_0^{x_2} (x_2 - s_2)^{\alpha_2-1} |b_{11}(x_1, s_2)| ds_2,$$

$$(\tilde{A}_2b_{11})(x) = \frac{\|a_2\|_{C(D)}}{\Gamma(\alpha_1)} \int_0^{x_1} (x_1 - s_1)^{\alpha_1-1} |b_{11}(s_1, x_2)| ds_1,$$

$$(\tilde{A}_3b_{11})(x) = \frac{\|a_3\|_{C(D)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1}\int_0^{x_2} (x_1 - s_1)^{\alpha_1-1}(x_2 - s_2)^{\alpha_2-1} |b_{11}(s_1, s_2)| ds_1 ds_2.$$

It is obvious that for any $n \in N$ the inequalities

$$\begin{aligned} |(A_1^n b_{11})\ (x)| &\le \left(\tilde{A}_1^n b_{11}\right)\ (x), \\ |(A_2^n b_{11})\ (x)| &\le \left(\tilde{A}_2^n b_{11}\right)\ (x), \\ |(A_3^n b_{11})\ (x)| &\le \left(\tilde{A}_3^n b_{11}\right)\ (x), \end{aligned} \tag{16}$$

are fulfilled in the each point $x \in D$.

Lemma 3. *For every natural number $n \in N$ is valid*

$$\begin{aligned} (\tilde{A}_1^n b_{11})(x) &= \frac{\left(\|a_1\|_{C(D)}\right)^n}{\Gamma(n\alpha_2)} \int_0^{x_2} (x_2 - s_2)^{n\alpha_2-1} |b_{11}(x_1, s_2)| ds_2, \\ (\tilde{A}_2^n b_{11})(x) &= \frac{\left(\|a_2\|_{C(D)}\right)^n}{\Gamma(n\alpha_1)} \int_0^{x_1} (x_1 - s_1)^{n\alpha_1-1} |b_{11}(s_1, x_2)| ds_1, \end{aligned} \tag{17}$$

$$(\tilde{A}_3^n b_{11})(x) = \frac{\left(\|a_3\|_{C(D)}\right)^n}{\Gamma(n\alpha_1)\Gamma(n\alpha_2)} \int_0^{x_1}\int_0^{x_2} (x_1 - s_1)^{n\alpha_1-1}(x_2 - s_2)^{n\alpha_2-1} |b_{11}(s_1, s_2)| ds_1 ds_2.$$

Proof. We prove the first statement of (17) by the induction method. First, for $n = 1$, we have:

$$(\tilde{A}_1b_{11})(x) = \frac{\|a_1\|_{C(D)}}{\Gamma(\alpha_2)} \int_0^{x_2} (x_2 - s_2)^{\alpha_2-1} |b_{11}(x_1, s_2)| ds_2,$$

which is true.

Next, we assume that the formula is true for $n = k$:

$$(\tilde{A}_1^k b_{11})(x) = \frac{\left(\|a_1\|_{C(D)}\right)^k}{\Gamma(k\alpha_2)} \int_0^{x_2} (x_2 - s_2)^{k\alpha_2-1} |b_{11}(x_1, s_2)| ds_2.$$

Assuming this, we must prove that the formula is true for its successor, $n = k + 1$. That is, we must show:

$$(\tilde{A}_1^{k+1}b_{11})(x) = \left(\tilde{A}_1(\tilde{A}_1^k b_{11})\right)(x) = \frac{\|a_1\|_{C(D)}}{\Gamma(\alpha_2)} \int\limits_0^{x_2} (x_2 - s_2)^{\alpha_2 - 1} \frac{\left(\|a_1\|_{C(D)}\right)^k}{\Gamma(k\alpha_2)}$$

$$\times \int\limits_0^{s_2} (s_2 - \tau_2)^{k\alpha_2 - 1} |b_{11}(x_1, \tau_2)|\, d\tau_2 ds_2 = \frac{\left(\|a_1\|_{C(D)}\right)^{k+1}}{\Gamma(\alpha_2)\Gamma(k\alpha_2)}$$

$$\times \int\limits_0^{x_2} |b_{11}(x_1, s_2)| \int\limits_{s_2}^{x_2} (x_2 - \tau_2)^{\alpha_2 - 1}(\tau_2 - s_2)^{k\alpha_2 - 1} d\tau_2 ds_2$$

$$= \frac{\left(\|a_1\|_{C(D)}\right)^{k+1}}{\Gamma(\alpha_2)\Gamma(k\alpha_2)} \int\limits_0^{x_2} |b_{11}(x_1, s_2)|\ (x_2 - s_2)^{(k+1)\alpha_2 - 1} B(\alpha_2, k\alpha_2) ds_2$$

$$= \frac{\left(\|a_1\|_{C(D)}\right)^{k+1}}{\Gamma((k+1)\alpha_2)} \int\limits_0^{x_2} (x_2 - s_2)^{(k+1)\alpha_2 - 1} |b_{11}(x_1, s_2)|\ ds_2,$$

where

$$B(a, b) = \int\limits_0^1 x^{a-1}(1 - x)^{b-1} dx,\ \ a,\, b > 0.$$

Thus the first formula of (17) is therefore true for each natural number. By analogy, it can be easily seen that second and third statements of (17) can be proved. □

For the simplification of the notation let us consider the unit sphere

$$S = \left\{b_{11} :\ \|b_{11}\|_{C(D)} = 1\right\}.$$

Using Lemma 3, with the help of the induction method the following lemma is proved.

Lemma 4. *For every natural number $n \in N$ and an arbitrary function $b_{11}(x) \in S$ the following inequalities hold:*

$$\begin{array}{l} (\tilde{A}_1^n b_{11})(x) \le \frac{\left(\|a_1\|_{C(D)} x_2^{\alpha_2}\right)^n}{\Gamma(n\alpha_2 + 1)},\ x \in D, \\ (\tilde{A}_2^n b_{11})(x) \le \frac{\left(\|a_2\|_{C(D)} x_1^{\alpha_1}\right)^n}{\Gamma(n\alpha_1 + 1)},\ x \in D, \\ (\tilde{A}_3^n b_{11})(x) \le \frac{\left(\|a_3\|_{C(D)} x_1^{\alpha_1} x_2^{\alpha_2}\right)^n}{\Gamma(n\alpha_1 + 1)\Gamma(n\alpha_2 + 1)},\ x \in D. \end{array} \tag{18}$$

Lemma 5. *For the every operator A of the form (12) the following relation is valid*

$$\sum_{n=0}^{\infty} \|A^n\| < E_{\alpha}(3\delta) \cdot E_{\alpha_1}(3\delta) \cdot E_{\alpha_2}(3\delta), \tag{19}$$

where $\|\cdot\|$ is a standard norm in the space of the linear operators, acting in the space $C(D)$, $\delta = \max\left(\|a_1\|_{C(D)} X_2^{\alpha_2},\ \|a_2\|_{C(D)} X_1^{\alpha_1}, \|a_3\|_{C(D)} X_1^{\alpha_1} X_2^{\alpha_2}\right)$, $E_{\alpha}(3\delta) = \sum\limits_{k=0}^{\infty} \frac{(3\delta)^k}{\Gamma(\alpha_1 k + 1) \cdot \Gamma(\alpha_2 k + 1)}$ and $E_{\alpha_i}(3\delta) = \sum\limits_{k=0}^{\infty} \frac{(3\delta)^k}{\Gamma(\alpha_i k + 1)}$, $i = 1,\, 2$ are Mittag-Leffler functions.

Proof. Note, that the operators $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ are in pairs commutative. Therefore with the help of the induction method one can prove that the following inequality is correct for any function $b_{11}(x) \in C(D)$:

$$|((A_1 + A_2 + A_3)^n b_{11})\ (x)| \le \left((\tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3)^n b_{11}\right)\ (x)$$

$$= \left(\left(\sum_{\substack{0 \le i,j,k \le n \\ i+j+k=n}} \frac{n!}{i!\,j!\,k!} \tilde{A}_1^i \tilde{A}_2^j \tilde{A}_3^k\right) b_{11}\right)\ (x).$$

Using estimates of Lemma 4 for all functions $b_{11}(x) \in S$, we have:

$$\left(\left(\sum_{\substack{0 \le i,j,k \le n \\ i+j+k=n}} \frac{n!}{i!\,j!\,k!} \tilde{A}_1^i \tilde{A}_2^j \tilde{A}_3^k\right) b_{11}\right)\ (x)$$

$$\le \sum_{\substack{0 \le i,j,k \le n \\ i+j+k=n}} \frac{n!}{i!\,j!\,k!} \frac{\left(\|a_3\|_{C(D)} X_1^{\alpha_1} X_2^{\alpha_2}\right)^k}{\Gamma(k\alpha_1+1)\Gamma(k\alpha_2+1)} \frac{\left(\|a_2\|_{C(D)} X_1^{\alpha_1}\right)^j}{\Gamma(j\alpha_1+1)} \frac{\left(\|a_1\|_{C(D)} X_2^{\alpha_2}\right)^i}{\Gamma(i\alpha_2+1)}.$$

Therefore

$$|((A_1 + A_2 + A_3)^n b_{11})\ (x)| \le \sum_{\substack{0 \le i,j,k \le n \\ i+j+k=n}} \frac{n!}{i!\,j!\,k!} \frac{\left(\|a_3\|_{C(D)} X_1^{\alpha_1} X_2^{\alpha_2}\right)^k}{\Gamma(k\alpha_1+1)\Gamma(k\alpha_2+1)}$$

$$\times \frac{\left(\|a_2\|_{C(D)} X_1^{\alpha_1}\right)^j}{\Gamma(j\alpha_1+1)} \frac{\left(\|a_1\|_{C(D)} X_2^{\alpha_2}\right)^i}{\Gamma(i\alpha_2+1)}.$$

Then

$$\|(A_1 + A_2 + A_3)^n\|_{C(D)} = \max_{\substack{b_{11}(x) \in S \\ x \in D}} |(A_1 + A_2 + A_3)^n b_{11}|\ (x)$$

$$\le \sum_{\substack{0 \le i,j,k \le n \\ i+j+k=n}} \frac{n!}{i!\,j!\,k!} \frac{\delta^k}{\Gamma(k\alpha_1+1)\Gamma(k\alpha_2+1)} \frac{\delta^j}{\Gamma(j\alpha_1+1)} \frac{\delta^i}{\Gamma(i\alpha_2+1)}.$$

Hence it follows that the estimate

$$\sum_{n=0}^{\infty} \|A^n\| \le E_\alpha(3\delta) \cdot E_{\alpha_1}(3\delta) \cdot E_{\alpha_2}(3\delta),$$

is true. $\square$

Using inequality (19) we have

$$\left\|(I+A)^{-1}\right\| \leq \left\|\sum_{n=0}^{\infty} A^n\right\| \leq \sum_{n=0}^{\infty} \|A^n\| \leq E_{\alpha}(3\delta)\cdot E_{\alpha_1}(3\delta)\cdot E_{\alpha_2}(3\delta)\,.$$

Therefore the operator $I+A$ is invertible in the space $C(D)$. Then from (15) we have

$$b_{11}+(I+A)^{-1}Bb_{11}=(I+A)^{-1}\Phi.$$

Now estimate the norm $\|B\|$ of the operator $B: C(D)\to C(D)$. It is obvious, that

$$\|B\| = \max_{\substack{b_{11}\in S\\ x\in D}} \left| -\frac{\beta_1 a_1(x)}{(\alpha_1+\beta_1)\Gamma(\alpha_2)}\int_0^{X_2}(X_2-s_2)^{\alpha_2-1}b_{11}(x_1,s_2)ds_2 \right.$$

$$-\frac{\beta_2 a_2(x)}{(\alpha_2+\beta_2)\Gamma(\alpha_1)}\int_0^{X_1}(X_1-s_1)^{\alpha_1-1}b_{11}(s_1,x_2)ds_1$$

$$-\frac{\beta_1 a_3(x)}{(\alpha_1+\beta_1)\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{x_1}\int_0^{X_2}(x_1-s_1)^{\alpha_1-1}(X_2-s_2)^{\alpha_2-1}b_{11}(s_1,s_2)ds_1ds_2$$

$$\left. -\frac{\beta_2 a_3(x)}{(\alpha_2+\beta_2)\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{x_2}\int_0^{X_1}(X_1-s_1)^{\alpha_1-1}(x_2-s_2)^{\alpha_2-1}b_{11}(s_1,s_2)ds_1ds_2 \right|$$

$$\leq \frac{|\beta_1|\,\|a_1\|_{C(D)}}{|\alpha_1+\beta_1|\,\Gamma(\alpha_2+1)}X_2^{\alpha_2} + \frac{|\beta_2|\,\|a_2\|_{C(D)}}{|\alpha_2+\beta_2|\,\Gamma(\alpha_1+1)}X_1^{\alpha_1}$$

$$+\frac{|\beta_1|\,\|a_3\|_{C(D)}\,x_1^{\alpha_1}}{|\alpha_1+\beta_1|\,\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}X_2^{\alpha_2} + \frac{|\beta_2|\,\|a_3\|_{C(D)}\,x_2^{\alpha_2}}{|\alpha_2+\beta_2|\,\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}X_1^{\alpha_1}$$

$$\leq \bar{k}_1\,\|a_1\|_{C(D)}\,X_2^{\alpha_2}\Gamma(\alpha_1+1)+\bar{k}_2\Gamma(\alpha_2+1)\,\|a_2\|_{C(D)}\,X_1^{\alpha_1}$$

$$+\bar{k}_1 x_1^{\alpha_1}X_2^{\alpha_2}\,\|a_3\|_{C(D)}+\bar{k}_2 X_1^{\alpha_1}x_2^{\alpha_2}\,\|a_3\|_{C(D)} \leq 4\bar{k}\delta,$$

where

$$\bar{k}_i=\frac{\beta_i}{|\alpha_i+\beta_i|\,\Gamma(\alpha_i+1)\Gamma(\alpha_j+1)},\, i,\, j=1,\ 2,\, i\neq j,$$

$$\bar{k}=\max(\bar{k}_1,\,\bar{k}_2).$$

Therefore, if

$$C=\left\|(I+A)^{-1}B\right\| \leq \left\|(I+A)^{-1}\right\|\ \|B\| \leq 4\bar{k}\delta\, E_{\alpha}(3\delta)\cdot E_{\alpha_1}(3\delta)\cdot E_{\alpha_2}(3\delta) < 1,$$

then integral equation (15) has a unique solution $b_{11}(x)\in C(D)$ for any $\Phi(x)\in C(D)$.

It is obvious, that under the condition $C<1$ the solution $b_{11}(x)\in C(D)$ of equation (15) satisfied also the condition

$$\|b_{11}\|_{C(D)} \leq \frac{1}{1-C}\left\|(I+A)^{-1}\right\|\ \|\Phi\|_{C(D)}\,. \tag{20}$$

Thus, the following theorem is valid.

Theorem 3. *If $\alpha_i + \beta_i \neq 0$, $i = 1, 2$ and $C < 1$, then equation (15) for any $\Phi(x) \in C(D)$ has a unique solution $b_{11}(x) \in C(D)$ satisfying (20).*

This theorem, states that equation (15) has a unique solution $b_{11} = P^{-1}y \in C(D)$ for any $y \in C(D)$, where $P = I + (I + A)^{-1}B$ and $y = (I + A)^{-1}\Phi$. By the use of this result, it can be easily proved that problem (1), (2) has a unique solution $u(x) \in C^{\alpha}(D)$ for every $\varphi = (\varphi_{11}(x), \varphi_1(x_1), \varphi_2(x_2), \varphi_0) \in H^{\alpha}$. If y is defined as $(I + A)^{-1}\Phi$, and $b_{11} = P^{-1}y$, then function (9) becomes a unique solution to problem (1), (2) in $C^{\alpha}(D)$. The estimation

$$\|u\|_{C^{\alpha}(D)} \leq K \cdot \|\varphi\|_{H^{\alpha}}, \quad K = const > 0, \tag{21}$$

is found from (9), (10) for the solution of (1), (2). Hence, the following theorem is proved:

Theorem 4. *If $\alpha_i + \beta_i \neq 0$, $i = 1, 2$ and $C < 1$, then (1), (2) is a well-posed problem, i.e. for each $\varphi(x) \in C^{\alpha}(D)$, there exists a unique solution $u(x) \in C^{\alpha}(D)$ that satisfies (21).*

Remark 1. *Note that for $\alpha_1 \neq 0$, $\beta_i = 0$, $i = 1, 2$ we have $C = 0 < 1$. Then by Theorem 4 (1), (2) is a well-posed problem.*

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